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Chaotic Iterative Algorithms for Image Reconstruction from Incomplete Projection Data

(Recommended by Prof. E. Dshalalow)

The problem of computer tomography is considered from incomplete projection data with chaotic algorithms for some particular systems of reconstruction. The numerical reconstruction algorithms and numerical simulation results are presented for a number of modeling objects which can be described by means of discrete functions.

Рассмотрена задача компьютерной томографии по неполным проекционным данным с использованием хаотических алгоритмов для некоторых специальных систем восстановления. Представлены численные алгоритмы восстановления и результаты численного моделирования для ряда моделируемых объектов, которые могут быть описаны посредством дискретных функций.

K e y   w o r d s : computer tomography, algebraic reconstruction technique (ART), asynchronous methods, chaotic algorithms, numerical simulation.

Technique of computerized tomography has a wide application not only in medicine but also for solving many technical problems. In these applications the projection data are often not available at each angle of view and may be very limited in number. In this case we say that we have an incomplete projection data. In particular, such kind of problems arises in mineral industries and engineering geophysics connected with acid drainage, the stability of mine workers, mineral exploration and others [1, 2]. For these problems the projection operator can be represented algebraically and the problem of image reconstruction is reduced to solving a system of linear algebraic equations. For solving such systems there often used different kinds of algebraic iterative algorithms the most well-known from which are Algebraic Reconstruction Technique (ART) algorithms [3—7]. They are generally simple, flexible and permit to use a priori knowledge of the object before its reconstruction that is very important in many practical applications.

In this paper we consider the problem of image reconstruction from incomplete projection data for some particular reconstruction systems which arise in some problems of engineering geophysics and mineral industry. For solving this problems we use the chaotic iterative algebraic algorithms. Numerical simulation of solving problems for image reconstruction from incomplete projection data for some modeling objects, comparing evaluations of errors and rate of convergence of these algorithms are presented. It is shown that for some choice of parameters we can obtain a good enough quality of reconstruction with these algorithms for some number of iterations.

Image reconstruction from incomplete projection data. Let $f(x, y)$ be a function which represents the spatial distribution of a physical parameter. If L is a line (ray) in the plane then the line integral

$$p_L = \int_L f(x, y) \, dL,$$

which is usually called a projection, is usually obtained from physical measurements.

From mathematical point of view the problem of reconstruction from projections is to find an unknown function $f(x, y)$ by means of a given set of projections p_L for all L. In practice we are given only discrete projection data that estimate p for a finite number of rays. And we want to find an image function $f(x, y)$, i. e. a reconstructed estimate of the unknown object. Moreover, we are deal with limited precision of measured data and so the projection data are given with some errors.

In many practical applications the projection data cannot be available for each direction and its number is very limited. In this case we say that we have a problem of image reconstruction with incomplete projection data. In particular, such kind of problems arise in mineral industries and engineering geophysics connected with acid drainage, the stability of mine workers, mineral exploration and others [1, 2].

In dependence on the obtaining system of projections there are many image reconstruction schemes, the main of them are parallel and beam schemes in the two-dimensional space. In some practical problems, in engineering for example, it is impossible to get projections from all directions because of the existing some important reasons (such as situation, size or impossibility of an access to a research object). This situation arises, for example, in the coal bed working (Fig. 1). In such a coal bed during the preparing process for working in dependence on the scheme the access to longwalls may be very difficult or impossible at all. Sometimes it is impossible to access to one or two sides of longwalls, and sometimes it is impossible only to access to the basis but all the longwalls are accessible. Each this situation has its own scheme of obtaining information.
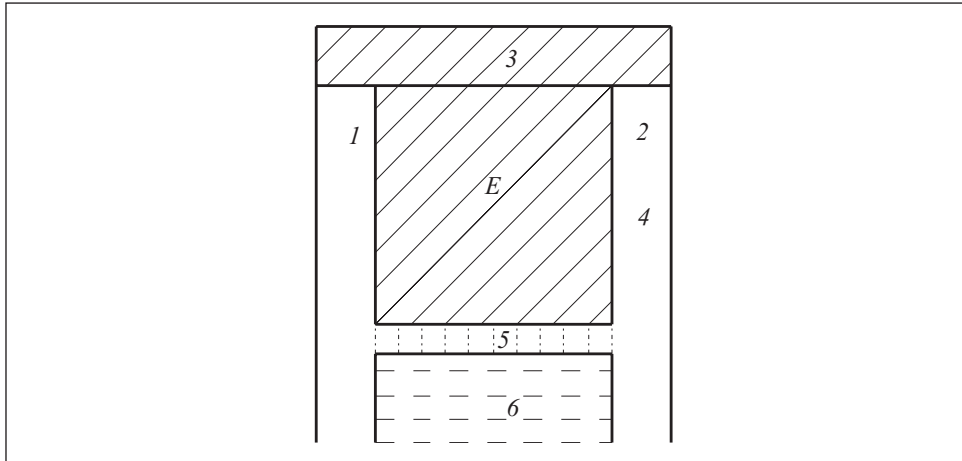
Fig. 1. The scheme of a coal bed working: *1* — sources; *2* — detectors; *3* — unmined coal; *4* — heading; *5* — longwall with mechanized lining, belt conveyor flight, heading machine so on; *6* – caving or filling; *E* – researching coal bed

In this paper we present results for image reconstructions only for two different schemes, which are described below.

Consider the scheme of obtaining projection data which we shall call as the system (1×1). In this scheme we have an access to a research coal bed from only two opposite sides. This situation often arises in engineering geophysics as well. In this case the sources of rays are situated only on one side and the detectors are situated on the opposite side of the research part of a coal bed. This scheme of obtaining information is shown in Fig. 2, *a*. And the second scheme of obtaining projection data, which we shall call the system (1×1, 1×1), is shown in Fig. 2, *b*. In this situation we can have an access to all four sides of a coal bed. Then the sources can been situated onto two neighboring sides, and the detectors can been situated on the opposite sides. So the projections can be obtained from two pair of the opposite sides.

Generalized model of asynchronous iterations. From mathematical point of view asynchronous algorithms are based on the method of asynchronous iterations which first introduced in the work [8], under the name "random relaxations" for solution of systems of linear algebraic equations. Further development of these methods and their generalization for the case of nonlinear operators was obtained in the work [9, 10]. The important results were obtained by El Tarazi [11] who introduced a visual model for the class of asynchronous algorithms, and obtained the first correct conditions of convergence in the nonlinear case for constraining operators.
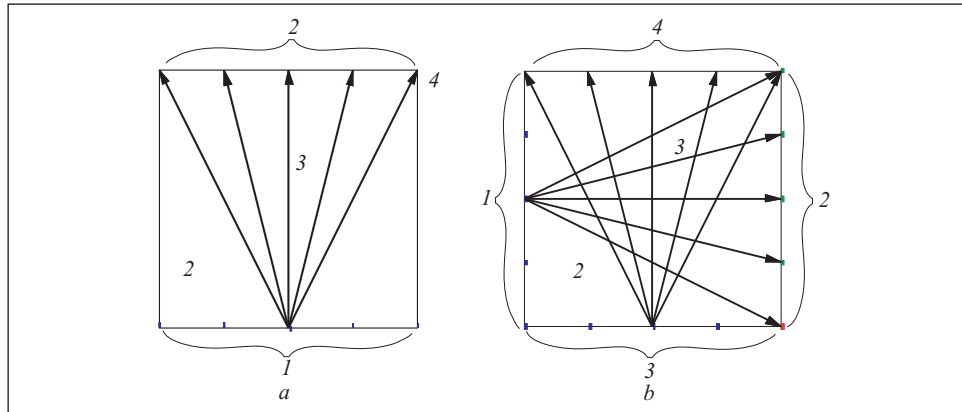
Fig. 2. The scheme of obtaining projection data: *a* — system (1×1); *b* — system (1×1, 1×1); *1* — sources of rays; *2* — research objects; *3* — rays; *4* — detectors

Let $T = \{\mathbf{T}_i\}_{i=1}^m$ be a set of nonlinear operators acting on the Euclidean space $\mathbf{R}^n$ and let $\mathbf{S}$ be an algorithmic operator. Consider the next iterative process:

$$\mathbf{y}^{k,i} = \mathbf{T}_i \mathbf{x}^{(k-1)}, \ \mathbf{x}^k = \mathbf{S}(\mathbf{x}^{(k-1)}, \{\mathbf{y}^{k,i}\}_{i=1}^m),$$

where $\mathbf{x}$ is an n-dimensional vector of the space $\mathbf{R}^n$, $i \in \{1, 2, ..., m\}$ for every $k = 0, 1, 2, ...$.

We will consider the parallel asynchronous implementation of such an iterative process on a parallel multiprocessor structure consisted of m independent elementary processors and some central processor. Each i-th elementary processor executes its calculations with its own pace in accordance with some formula correspondent to the operator $\mathbf{T}_i$. It has its own local memory and connects only with central processor. We assume that each elementary processor can have access to the central processor at any time. After each cycle of calculations of a vector $\mathbf{y}^{k,i}$ the i-th processor sends this value to the central processor and loads from it the new value $\mathbf{x}^k$ as its new initial data.

The important notion in the theory of asynchronous iterations is the sequence of chaotic sets.

Definition 1. A sequence of nonempty subsets $I = \{I_k\}_{k=0}^{\infty}$ of the set $\{1, 2..., ..., m\}$ is a sequence of chaotic sets if $\limsup_{j \to \infty} I_j = \{1, 2, ..., m\}$ (another words, if each integer $j \in \{1, 2, ..., m\}$ appears in this sequence infinite many times).

For the first time such sequences were used in the work [9].

Definition 2. If $I_k = \{j_k\}$, for any k, in a sequence of chaotic sets $I = \{I_k\}_{k=0}^{\infty}$ of the set $\{1, 2..., m\}$, (i. e. each subset I_k consists of only one element), then such sequence is called acceptable (or admissible).

Suppose that Parallel Computing System (PCS) consists of *m* of processors working local independently. In this case the notion of the sequence of chaotic sets has a simple interpretation: it sets the time diagram of a work of each processor during non-synchronous work of PCS. So the subset I_k is the set of the numbers of those processors which access the central processor at the same time.

Note, that the definition of the sequence of chaotic sets can be given in the following equivalent form.

Lemma 1. Let $I = \{I_k\}_{k=0}^{\infty}$ be a sequence of nonempty subsets of the set $\{1,2..., m\}$. Then the following conditions are equivalent:

1) $\lim\sup\limits_{k \to \infty} I_k = \{1, 2, ..., m\}$;

2) the set $\{k \mid i \in I_k\}$ is unlimited for each $i = 1,2..., m$.

3) for each $j \in \mathbf{N}$ there exists $p(j) \in \mathbf{N}$ such that the following condition satisfies:

$$\bigcup_{i=j+1}^{j+p(j)} I_i = \{1, 2, ..., m\}. \tag{1}$$

The proof of this lemma one can find, for example, in [4].

For any sequence of chaotic sets the numbers $p(j)$ depends on a number *j*. In practice and for research of convergence of asynchronous implementations of iterative processes there are more important sequences of chaotic sets, for which these numbers do not depend on the number j.

Definition 3. If for a sequence of chaotic sets $I = \{I_k\}_{k=0}^{\infty}$ the numbers $p(j)$, defined by (1), do not depend on the choice of number j, i. e., $p(j) = T = \text{const}$ for each $j \in \mathbf{N}$, then such a sequence is called regular, and the number *T* is called the number of regularity of the sequence *I*.

Note, that this definition coincides with concept of a regular sequence, introduced in the paper [11] for the case of admissible sequence.

Other important concept in the theory of asynchronous iterations is the notion of a sequence of delays.

Definition 4. A sequence $J = \{\sigma(k)\}_{k=1}^{\infty}$ of *m*-dimensional vectors $\sigma(k) = \{\sigma_1(k), \sigma_2(k), ..., \sigma_m(k)\}$ with integer coordinates, satisfying   the following conditions:

$$0 \le \sigma_i(k) \le k - 1; \tag{2}$$

$$\lim_{k \to \infty} \sigma_i(k) = \infty \tag{3}$$

for each $i = 1, 2, ..., m$ and $k \in \mathbf{N}$, is called a sequence of delays. In the case, when instead of condition (3) it holds the following condition:

there exists a fixed number $L \in \mathbf{N}$ such that

$$k - \sigma_i(k) \le L \tag{4}$$

for each $k \in \mathbf{N}$ and $i = 1, 2, ..., m$, the sequence is called a sequence with limited delays and the number L is called a delay, or an asynchronous measure.

The sequence of delays determines the numbers of using iterations by each fixed processor, and the number L shows a depth of used iterations and actually reflects possibilities of the concrete computing system. For synchronous implementation of the iterative process the difference $k - \sigma_i(k)$ is equal to 0 for all $i = 1, 2, ..., m$ and $k \in \mathbf{N}$.

Recall the definition of generalized model of asynchronous computational process [12].

Definition 5. Let we have a set of nonlinear operators $\mathbf{T}_i \colon \mathbf{R}^n \to \mathbf{R}^n$, $i \in \{1, 2, ..., m\}$ and an initial value of vector $\mathbf{x}^0 \in \mathbf{R}^n$. A generalized model of the asynchronous iterations with limited delays for the set of operators $\mathbf{T}_i$, $i = 1, 2, ..., m$ is a method of building the sequence of vectors $\{\mathbf{x}^k\}_{k=0}^{\infty}$, which is given recursively by the following scheme:

$$\mathbf{y}^{k,i} = \begin{cases} \mathbf{T}_i \mathbf{x}^{(\sigma^i(k))}, & \text{if } i \in I_k; \\ \mathbf{y}^{k-1,i}, & \text{otherwise}; \end{cases} \qquad \mathbf{x}^{(k)} = \mathbf{S}(\mathbf{x}^{(k-1)}, \{\mathbf{y}^{k,i}\}_{i \in I_k}),$$

where $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1, 2, ..., m\}$ and $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ are sequences of limited delays ($i = 1, 2, ..., m$).

Chaotic iterative algorithms for image reconstruction. The full discrete model of the problem of image reconstruction is reduced to solving a system of linear algebraic equations of the form:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{p}, \tag{5}$$

where $\mathbf{A} \in \mathbf{R}^{m,n}$ is the matrix of coefficients; $\mathbf{x} = \{x_1, x_2, ..., x_n\}^T \in \mathbf{R}^n$ is the image vector; $\mathbf{p} = \{p_1, p_2, ..., p_m\}^T \in \mathbf{R}^m$ is the measurement vector of projection data.

This system has a few characteristics: it is a rectangular as a rule and it has a very large dimension. For solving this system it is often used different kind of algebraic iterative algorithms which are based on the Kaczmarz method, the most well-known of which are the additive algorithm ART [3—6]. These algorithms are very flexible and allow to apply different a priory information about object before its reconstruction which is especially very important when we have an incomplete projection data. The basic idea of these algorithms is to run through all equations cyclically with modification of the present estimate $\mathbf{x}^{(k)}$ in such a way that the present equation with index i is fulfilled. Denote by

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^i, \mathbf{x}) - p_i}{\|\mathbf{a}^i\|^2} \mathbf{a}^i; \quad \mathbf{P}_i^{\omega} = (1 - \omega)\mathbf{I} + \omega \mathbf{P}_i,$$

where $\mathbf{a}^i$ is the i-th row of a matrix $\mathbf{A}$, $0 < \omega < 2$ is a relaxation parameter.

A l g o r i t h m 1 (ART):

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector.

2. The k-th iteration is calculated in accordance with following scheme: $\mathbf{x}^{(k+1)} = \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k)}$ ($i = 1, 2, ..., m$), where $\mathbf{P}_i^{\omega_k}$ are operators defined by (10), ω_k are relaxation parameters, $0 < \omega_k < 2$ and $i(k) = k \,(\mathrm{mod}\,m) + 1$.

Algorithm 1 was investigated in [6], and it was used successfully in medicine. This algorithm is consequence, and moreover the numbers of operators in this algorithm are chosen by the cyclic way.

In practice the vector of projection data is given as a rule with some error. Therefore instead of a system of linear equation (5) we have a system of linear inequalities:

$$\mathbf{p} - \mathbf{e} \le \mathbf{A} \cdot \mathbf{x} \le \mathbf{p} + \mathbf{e} , \qquad (6)$$

where $\mathbf{e} = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$ is a non-negative vector. And we can consider that the vector $\mathbf{e}$ is given a priory and defines the errors of projection data. In this case we can introduce the following projection operators:

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{((\mathbf{a}^i, \mathbf{x}) - p_i - \varepsilon_i)^+ - (p_i - \varepsilon_i - (\mathbf{a}^i, \mathbf{x}))^+}{\left\| \mathbf{a}^i \right\|^2} \, \mathbf{a}^i , \qquad (7)$$

where

$$s^+ = \begin{cases} s, & \text{if } \quad s \ge 0; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\mathbf{P}_i^{\omega} = (1 - \omega)\, \mathbf{I} + \omega\, \mathbf{P}_i \qquad (8)$$

where $\mathbf{a}^i$ is the i-th row of a matrix $\mathbf{A}$, $0 < \omega < 2$ is a relaxation parameter.

For solving the system of inequalities (6) we use the following algorithm.

A l g o r i t h m 2 (ART-3):

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector.

2. The k-th iteration is calculated in accordance with such a scheme: $\mathbf{x}^{(k+1)} = \mathbf{C} \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k)}$ ($i = 1, 2, ..., m$), where $\mathbf{P}_i^{\omega_k}$ are operators defined by (7) and (8), $0 < \omega_k < 2$ are relaxation parameters, $i(k) = k \,(\mathrm{mod}\,m) + 1$, and $\mathbf{C}$ is a constraining operator.

Now we apply the generalized model of asynchronous iterations for implementation of algorithm ART on non-synchronous computer structure. In this case we obtain the following chaotic algorithm, where the numbers of operators are chosen by the chaotic way.

A l g o r i t h m 3:

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector.

2. The $k + 1$-th iteration is calculated in accordance with such a scheme:

$$\mathbf{y}^{k+1,i} = \begin{cases} \mathbf{P}_i^{\omega_k} \mathbf{x}^{(\sigma^i(k))}, \text{ if } i \in I_k; \\ \mathbf{y}^{k,i}, \text{ otherwise}; \end{cases}$$

$$\mathbf{x}^{(k+1)} = \sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k+1,i}, \quad (i = 1, 2, ..., m),$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (7) and (8), $0 < \omega_k < 2$ are relaxation parameters, γ_i^k are positive real numbers with property $\sum_{i \in I_k} \gamma_k^i = 1$ for each $k \in \mathbf{N}$, $I = $

$= \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1, 2, ..., m\}$, $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ are sequences of chaotic sets.

The convergence of this algorithm is given by the following theorem.

Theorem 1. Let system (5) be consistent, $I = \{I_k\}_{k=1}^{\infty}$ be a regular sequence of chaotic sets $I_k \subset \{1, 2, ..., m\}$ with a number of regularity T, $\{\sigma^i(k)\}_{k=1}^{\infty}$ be sequences with limited delays and $\sigma_j^i(k) = \sigma_i(k)$ and , and let the number of delay be equal to T. Then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ defined by algorithm 3 converges to some point $\mathbf{x}^* \in H$, which is a fixed point of orthogonal projection operators $\mathbf{P}_i$ ($i = 1, 2, ..., m$).

The full proof of this theorem one can find in [4]. We now consider the particular case of algorithm 2, Chaotic Algebraic Reconstruction Technique (CHART) when we have no delays and the sequence of chaotic sets is acceptable.

A l g o r i t h m 4 (CHART-3):

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector.

2. The $k+1$-th iteration is calculated in accordance with following scheme:

$$\mathbf{y}^{k,i} = \begin{cases} \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k-1)}, \text{ if } i \in I_k, \\ \mathbf{y}^{k-1,i}, \text{ otherwise}; \end{cases}$$

$$\mathbf{x}^{(k)} = \mathbf{C} \sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k,i}, \quad (i = 1, 2, ..., m),$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (7) and (8), $0 < \omega_k < 2$, are relaxation parameters, γ_i^k are positive real numbers with property $\sum_{i \in I_k} \gamma_k^i = 1$ for each $k \in \mathbf{N}$,

$I = \{I_k\}_{k=1}^{\infty}$ is an acceptable sequence of chaotic sets such that $I_k \subset \{1, 2, ..., m\}$ and **C** is a constraining operator. In this paper we consider that **C**$= C_1 \, C_2 \, C_3$ where

$$C_1[\mathbf{x}] = \begin{cases} \mathbf{x}, & \text{if} \quad \mathbf{x} \in E, \\ 0, & \text{otherwise}; \end{cases} \qquad (C_2[\mathbf{x}])_i = \begin{cases} a, & \text{if} \quad x_i < a, \\ x_i, & \text{if} \quad a \leq x_i \leq b, \\ b, & \text{if} \quad x_i > b; \end{cases}$$

$$(C_3[\mathbf{x}])_j = \begin{cases} 0, & \text{if} \quad p_i = 0 \wedge a_{ij} \neq 0, \\ x_j, & \text{otherwise}. \end{cases}$$

The convergence of algorithm 4 is given by the following theorem.

Theorem 2. Let system (6) be consistent, $I = \{I_k\}_{k=1}^{\infty}$ be an acceptable sequence of chaotic sets $I_k \subset \{1, 2, ..., m\}$. Then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ defined by algorithm 4 converges to some solution of this system.

The proof of this theorem is the same as theorem 1.

Computer simulation and experimental results. In order to evaluate the goodness of the compute reconstruction of a high-construct image from incomplete data we tested different kinds of geometric figures and reconstruction schemes.

An important factor in the simulation process of image reconstruction is the choice of modeling objects which describe the density distribution of research objects. In a coal bed, where we search the reservoirs of compressed gas or interlayers of a barren rock, the density distribution may be considered discrete and the density difference of these three environments (coal, compressed gas and barren rock) is significant. Therefore for illustration of the implementation of the algorithms working with incomplete data we chose the discrete functions with high contrast. In this paper we present only the results of computer reconstruction for two discrete figures and two reconstructions systems (1×1) and ($1 \times 1, 1 \times 1$).

The first discrete function $f_1(x, y)$ is given by the following equation:

$$f_1(x, y) = \begin{cases} 1, & (x, y) \in D \subset E \subset \mathbf{R}^2, \\ 0, & \text{otherwise}, \end{cases}$$

where E is a square $E = \{(x, y): -1 \leq x, y \leq 1\}$, and D is a subset of E of the form $D = = [-0.4, -0.2] \times [-0.5, 0.5] \cup [-0.2, 0.2] \times [0.3, 0.5] \cup [-0.2, 0.2] \times [-0.1, 0.1] \cup \cup [0, 0.2] \times [0.1, 0.3]$.
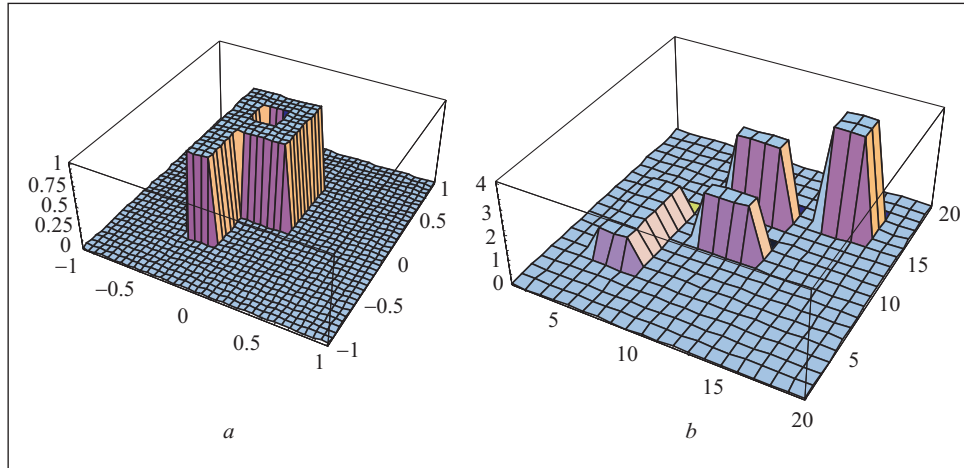
Fig. 3. The original functions $f_1(x, y)\,(a)$ and $f_2(x, y)\,(b)$

The second figure is given by the following equation:

$$f_2(x,y)=\begin{cases}1, & (x,y)\in D_1\subset E\subset \mathbf{R}^2, \\ 2, & (x,y)\in D_2\subset E\subset \mathbf{R}^2, \\ 3, & (x,y)\in D_3\subset E\subset \mathbf{R}^2, \\ 4, & (x,y)\in D_4\subset E\subset \mathbf{R}^2, \\ 0, & \text{otherwise},\end{cases}$$

where E is a square $E=\{(x,y): -1\leq x,y\leq 1\}$, and D_i are subsets of E of the form

$$D_1 = [-0.7, -0.4] \times [-0.5, 0.2], \quad D_2= [-0.2, 0.2] \times [-0.1, 0.1],$$

$$D_3 = [-0.2, 0.2] \times [0.3, 0.5], \quad D_4 = [0.4, 0.7] \times [0.4, 0.7].$$

The three dimensional view of the plots of functions $f_1(x, y)$ and $f_2(x, y)$ are presented in Fig. 3. As was shown earlier (see for example [3, 5, 12]), the image reconstruction of such objects from complete data gives a good enough results after 6-7 full iterations.

In this paper we present the numerical results of image reconstruction of chosen functions with algebraic iterative algorithms ART-3 and chaotic algorithm CHART-3 for two reconstruction systems (1×1) and (1×1, 1×1). We compare these results of reconstructions, and we investigate the influence of various parameters of these algorithms such as a pixel initialization, relaxation parameters, number of iterations and noise in the projection data on reconstruction
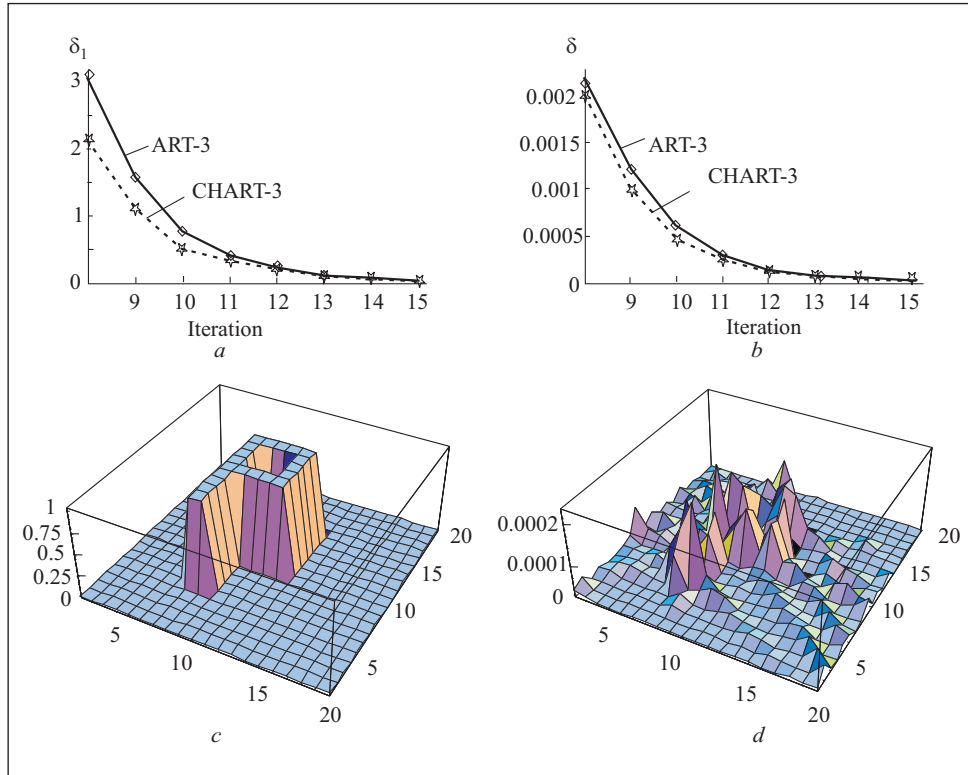
Fig. 4. The numerical simulation results of image reconstruction of $f_1(x, y)$ with algorithms ART-3 and CHART-3 for $n = 20 \times 20$, $m = 644$ in the system $(1 \times 1, 1 \times 1)$

quality. The convergence of these algorithms was studied in dependence on different these parameters. The convergence characteristic plots are given in view of plots for the maximum relative error:

$$\delta_1 = \frac{\max_i \left| f_i - \widetilde{f}_i \right|}{\max_i \left| f_i \right|} 100\,\%,$$

and the mean absolute error

$$\delta = \frac{1}{n} \sum_i \left| f_i - \widetilde{f}_i \right|,$$

where f_i is the value of a given modeling function in the center of the i-th pixel and $\widetilde{f}_i$ is the value of the reconstructed function in the i-th pixel.

The reconstruction results of $f_1(x, y)$ with algorithms ART-3 and CHART-3 in the reconstruction scheme $(1 \times 1, 1 \times 1)$ for $n = 20 \times 20$ pixels, $m = 644$ projections
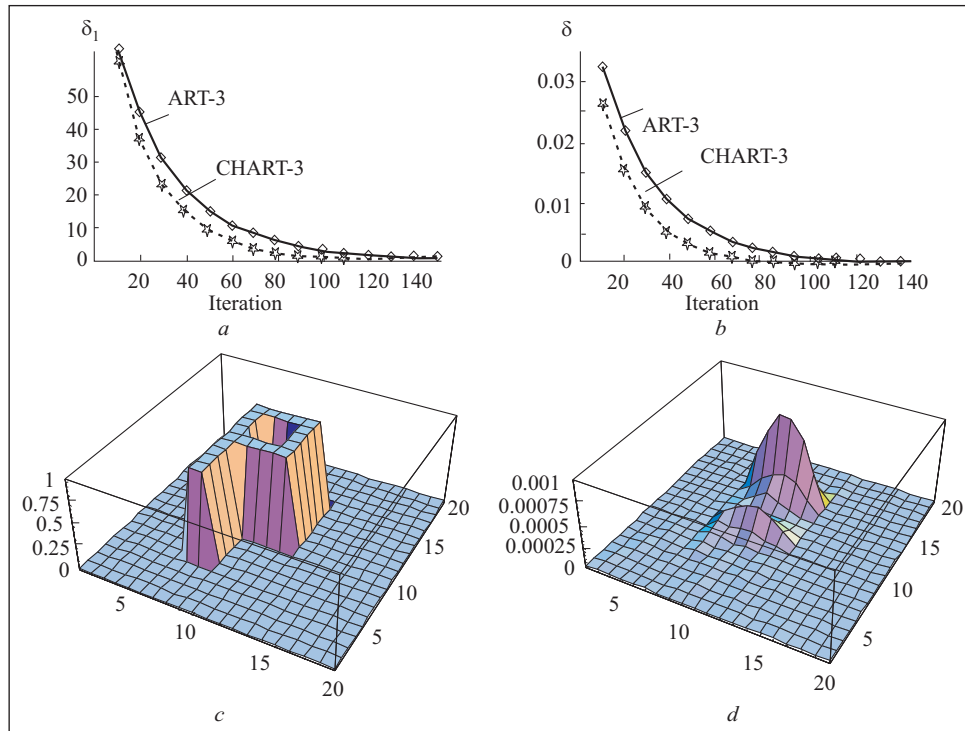
Fig. 5. The numerical simulation results of image reconstruction of $f_1(x, y)$ with ART-3 and CHART-3 for $n = 20 \times 20$, $m = 788$ in the system $(1 \times l)$

are presented in Fig. 4. The plots, which are presented in Fig. 4, *a*, *b*, illustrate the dependence of the maximum relative error and the mean absolute error on number of iterations, correspondingly. The three dimensional view of the plot of the reconstruction function $f_1(x, y)$ by algorithm ART-3 after 15 iterations is shown in Fig. 4, *c*, and the plot of the mean absolute errors for this image reconstruction is shown in Fig. 4, *d*.

The same function $f_1(x, y)$ was also reconstructed in the system $(1 \times l)$. The numerical simulation results of this reconstruction with algorithms ART-3 and CHART-3 for $n = 20 \times 20$, $m = 788$ are shown in Fig. 5. The plots, which are presented in Fig. 5, *a*, *b*, illustrate the dependence of the maximum relative error and the mean absolute error on the number of iterations, correspondingly. The three dimensional view of the plot of the reconstruction function $f_1(x, y)$ by algorithm CHART-3 after 100 iterations is shown in Fig. 5, *c*, and the plot of the mean absolute errors for this image reconstruction is shown in Fig. 5, *d*.

The analogous reconstructions were obtained for the function $f_2(x, y)$. The numerical simulation results of image reconstructions for $f_2(x, y)$ with algo-
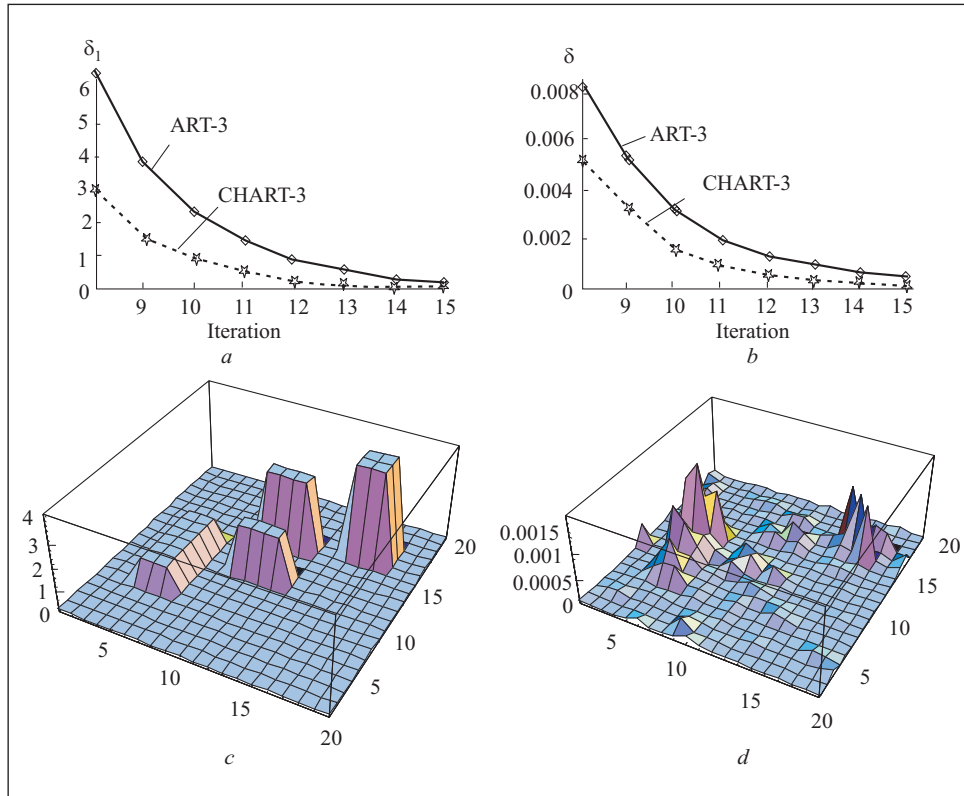
Fig. 6. The numerical simulation results of image reconstruction of $f_2(x, y)$ with ART-3 and CHART-3 for $n = 20 \times 20$, $m = 644$ in the system $(1 \times 1, 1 \times 1)$.

rithms ART-3 and CHART-3 in the system $(1 \times 1, 1 \times 1)$ for $n = 20 \times 20$ pixels, $m = 644$ projections are given in Fig. 6. The plots, which are presented in Fig. 6, *a*, *b*, illustrate the dependence of the maximum relative error and the mean absolute error on number of iterations, correspondingly. The three dimensional view of the plot of the reconstruction function $f_2(x, y)$ by algorithm CHART-3 after 15 iterations is shown in Fig. 6, *c*, and the plot of the mean absolute errors for this image reconstruction is shown in Fig. 6, *d*.

The same function $f_2(x, y)$ was also reconstructed in the system (1×1). The numerical simulation results of this reconstruction with algorithms ART-3 and CHART-3 for $n = 20 \times 20$, $m = 788$ are shown in Fig. 7. The plots, which are presented in Fig. 7, *a*, *b*, illustrate the dependence of the maximum relative error and the mean absolute error on number of iterations, correspondingly. The three dimensional view of the plot of the reconstruction function $f_2(x, y)$ by algorithm CHART-3 after 25 iterations is shown in Fig. 7, *c*, and the plot of the mean absolute errors for this image reconstruction is shown in Fig. 7, *d*.
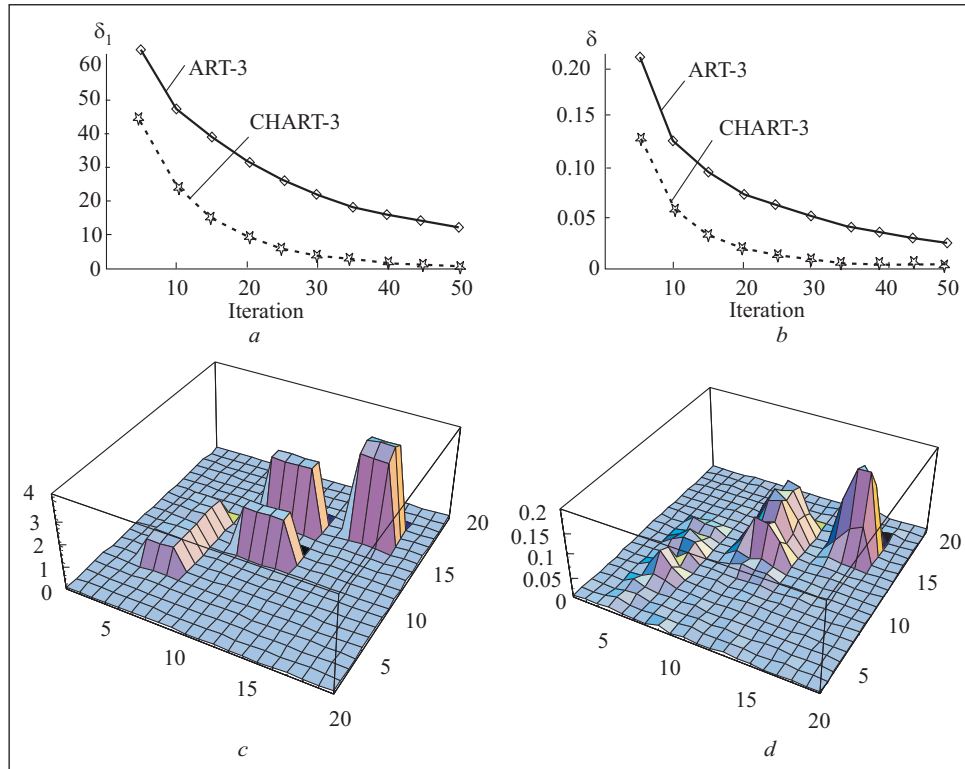
Fig. 7. The numerical simulation results of image reconstruction of $f_1(x, y)$ with ART-3 and CHART-3 for $n = 20 \times 20$, $m = 788$ in the system (1× l)

All algorithms were implemented on IBM/PC (processor AMD Duron XP, 1600 MHz) by means of C++ and MATHEMATICA 5.1. One iteration by means of Mathematica 5.1 was implemented approximately 0.5s for algorithm ART-3 and CHART-3, and in C++ one iteration for both algorithms is implemented in a real time.

Conclusion. The aim of this paper was an elaboration and comparison of chaotic iterative algorithms for reconstruction of high-contrast objects from incomplete projection data. We study the quality and convergence of these algorithms. The experimental results show that convergent characteristics of algorithm CHART-3 are considerably better by comparison with algorithm ART-3. And the configuration (1×1, 1×1) is considerably better by comparison with system (1×l). For each considered system of reconstruction there exist the parameters which allow to obtain an enough good quality of reconstruction after the number of iterations but this number is considerably larger than for reconstruction with complete projection data. The number of iterations for achieving the

stable reconstruction is approximately two times more for the second system by comparison with the first one. And this number is approximately 10 times more for the system (1×1, 1×l) by comparison with the case of the complete data.

Розглянуто задачу комп'ютерної томографії по неповним проективним даним з використанням хаотичних алгоритмів для деяких спеціальних систем відновлення. Наведено числові алгоритми відновлення та результати числового моделювання для об'єктів, які можуть бути описані дискретними функціями.

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